

# A generalization of boson normal ordering

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In this paper we define generalizations of boson normal ordering. These are based on the number of contractions whose vertices are next to each other in the linear representation of the boson operator function. Our main motivation is to shed further light onto the combinatorics arising from algebraic and Fock space properties of boson operators.

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## I. INTRODUCTION

Generalizations and refinements of normal ordering have been studied in the literature (see [1, 2, 3, 4]). Here we introduce a generalization of the normal ordering for boson operator functions. The generalization is based on the number of contractions whose vertices are next to each other in the linear representation of the function itself. In this way, we introduce a parameter associated to these specific contractions. The use of the parameter allows to refine the set of contractions and to give a statistics on this set, obtaining the standard normal ordering as a special case. Additionally, we outline a framework for an even more general approach to normal orderings, where contractions between vertices of different distances are allowed to have arbitrary weights. With this tools we are able to extend some of the results of Katriel [1], where a concise theory of Bell polynomials was described in terms of the algebraic and Fock space properties of the boson operators. Keeping in mind that general formulas for normally ordered forms are commonly difficult to obtain (see [5]), our analysis is potentially useful in computing the standard normal ordering, since it provides a larger perspective, at least from the combinatorial point of view.

The paper is organized as follows. In Section II we introduce a generalization of the boson normal ordering. In Section III we discuss some physical aspects of this generalization. In Section IV we give an explicit formula for the generalized normally ordered form of the function  $(a^\dagger a)^n$  (Theorem 5). In order to establish the result, we prove that there exists a bijection between the set of contractions of  $(a^\dagger a)^n$  having degree  $k$  and the set of partitions of  $n$  with  $k$  rises (Proposition 3). Considering coherent states (see, *e.g.*, [9]), we prove relations between the generalized normal ordering and integer sequences like the Bell numbers and Stirling numbers, thus generalizing observations of [1]. Finally, in Section V we describe a rather general framework for generalizations of normal ordering by introducing weights for contractions satisfying some natural conditions. However, on this general level we neither consider explicit expressions nor physical applications; this will be left for future investigations.

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## II. A GENERALIZATION OF THE BOSON NORMAL ORDERING

Let  $b$  and  $b^\dagger$  be the boson annihilation and creation operators satisfying the commutation relation

$$[b, b^\dagger] \equiv bb^\dagger - b^\dagger b = I. \quad (1)$$

The *normal ordering* is a functional representation of boson operator functions in which all the creation operators stand to the left of the annihilation operators. A function  $F(b, b^\dagger)$  can be seen as a word (of possible infinite length) on the alphabet  $\{b, b^\dagger\}$ . We denote by  $\mathcal{N}[F(b, b^\dagger)]$  the normal ordering of a function  $F(b, b^\dagger)$ . We can obtain  $\mathcal{N}[F(b, b^\dagger)]$  from  $F(b, b^\dagger)$  by means of contractions and double dot operations. The *double dot operation* deletes all the letters  $\emptyset$  and  $\emptyset^\dagger$  in the word and then arranges it such that all the letters  $b^\dagger$  precede the letters  $b$ . For example,  $:b^k(b^\dagger)^l:= (b^\dagger)^l b^k$ . A *contraction* consists of substituting  $b = \emptyset$  and  $b^\dagger = \emptyset^\dagger$  in the word whenever  $b$  precedes  $b^\dagger$ . Among all possible contractions, we also include the null contraction, that is the contraction leaving the word as it is. Specifically,

$$\mathcal{N}[F(b, b^\dagger)] = F(b, b^\dagger) = \sum : \{\text{all possible contractions}\} : . \quad (2)$$

For example, the normal ordering of the word  $bb^\dagger bbb^\dagger bb$  is  $(b^\dagger)^2 b^5 + 4b^\dagger b^4 + 2b^3$ .

Several authors established different connections between Stirling and Bell numbers and normally ordered forms (for example, see [1, 2] and references therein). Consider the number operator  $N = b^\dagger b$ . The normally ordered form of its  $n$ -th power can be written as

$$(b^\dagger b)^n = \sum_{k=1}^n S(n, k) (b^\dagger)^k b^k, \quad (3)$$

where the integers  $S(n, k)$  are the so called Stirling numbers of second kind (see [6, Seq. A008277]). One may also define the so called Bell polynomials  $B(n, x) = \sum_{k=1}^n S(n, k) x^k$  and Bell numbers  $B(n) = B(n, 1) = \sum_{k=1}^n S(n, k)$ . The Stirling numbers satisfy the following recurrence relation

$$S(n+1, k) = kS(n, k) + S(n, k-1) \quad (4)$$

with the initial conditions  $S(n, 0) = \delta_{n,0}$  and  $S(n, k) = 0$  for  $k > n$  [6].

Contractions can be depicted with diagrams called linear representations. Let us consider a word  $\pi$  on the alphabet  $\{b, b^\dagger\}$  of length  $n$ , i.e.,  $\pi = \pi_n \cdots \pi_1$  with  $\pi_i \in \{b, b^\dagger\}$ . We draw  $n$  vertices, say  $1, 2, \dots, n$ , on a horizontal line, such that the point  $i$  corresponds to the letter  $\pi_i$ ; we represent each  $b$  by a white vertex and each letter  $b^\dagger$  by a black vertex; a black vertex  $i$  can be connected by an undirected edge  $(i, j)$  to a white vertex  $j$  (but there may also be black vertices having no edge). Importantly, the edges are drawn in the plane above the points. This is the *linear representation* of a contraction. An example is given in Figure 1 for the word  $bbb^\dagger b^\dagger$ .

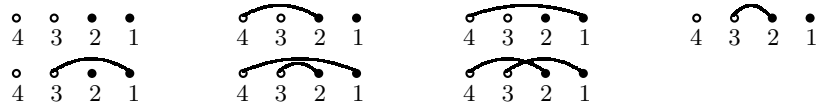


FIG. 1: The linear representation of the contractions of the word  $bbb^\dagger b^\dagger$

Now let us generalize the definition of normal ordering as given above. For this we consider the alphabet consisting of two letters  $a$  and  $a^\dagger$  but where now these two letters are not assumed to satisfy any commutation relation (like, e.g., (1)); to avoid confusion with the bosonic operators satisfying (1) we have chosen different letters here. Let  $F(a, a^\dagger)$  be a possibly infinite word on the alphabet  $\{a, a^\dagger\}$ .

We define  $\mathcal{C}(F(a, a^\dagger))$  to be the multiset of all words obtained by substituting  $a = e$  and  $a^\dagger = e^\dagger$  whenever  $a$  precedes  $a^\dagger$ ; moreover, we replace any two adjacent letters  $e$  and  $e^\dagger$  with  $p$ . For example,  $\mathcal{C}(aaa^\dagger a^\dagger) = \{aaa^\dagger a^\dagger, eae^\dagger a^\dagger, eaa^\dagger e^\dagger, apa^\dagger, aea^\dagger e^\dagger, epe^\dagger, eee^\dagger e^\dagger\}$ , as it is illustrated in Figure 1. For each word  $\pi$  in  $\mathcal{C}(F(a, a^\dagger))$  the double dot operation of  $\pi$  is defined by deleting all letters  $e$  and  $e^\dagger$  and arranging it such that all letters  $a^\dagger$  precede the letter  $a$ ; clearly,  $:\pi := (a^\dagger)^v a^u p^w$  for some  $u, v, w \geq 0$ . We now define

$$\mathcal{N}_p[F(a, a^\dagger)] := \sum_{\pi \in \mathcal{C}(F(a, a^\dagger))} : \pi : . \quad (5)$$

For example,

$$\begin{aligned} \mathcal{N}_p((a^\dagger a)^3) &= : a^\dagger a a^\dagger a a^\dagger a : + : a^\dagger a a^\dagger p a : + : a^\dagger e a^\dagger a e^\dagger a : + : a^\dagger p a a^\dagger a : + : a^\dagger p p a : \\ &= (a^\dagger)^3 a^3 + (2p + 1)(a^\dagger)^2 a^2 + p^2 a^\dagger a. \end{aligned} \quad (6)$$

Clearly  $\mathcal{N}_p(F(a, a^\dagger))$  is a generalization of the normally ordered form  $\mathcal{N}(F(a, a^\dagger))$ , namely  $\mathcal{N}(F(a, a^\dagger)) = \mathcal{N}_1(F(a, a^\dagger))$ .

Notice that, for all  $v, u \geq 0$ ,

$$\mathcal{N}_p(a^v (a^\dagger)^u) = \sum_{i=0}^v \left[ \binom{v}{i} (u-i) + (v-1+p) \binom{v-1}{i-1} \right] (u-1)_{i-1} (a^\dagger)^{u-i} a^{v-i}$$

where  $x^{\underline{i}} = x(x-1)\cdots(x-i+1)$  and  $x^{\overline{i}} = (x+1)(x+2)\cdots(x+i)$  with  $x^{\underline{0}} = 1$ . In fact,

$$\mathcal{N}_p(a^v (a^\dagger)^u) = a^\dagger \mathcal{N}_1(a^v (a^\dagger)^{u-1}) + (v-1+p) \cdot \mathcal{N}_1(a^{v-1} (a^\dagger)^{u-1}).$$

### III. SOME PHYSICAL CONSIDERATIONS CONCERNING $\mathcal{N}_p$

Before deriving an explicit expression for  $\mathcal{N}_p((a^\dagger a)^n)$  in the next section, we want to discuss some more physical aspects of the generalized normal ordering. Recall that in the usual case the commutation relations of the bosonic operators are used to derive the normal ordered form of a function of annihilation and creation operators in the form of *Wick's theorem*, cf. p.159 in [7] (there also exists a version for fermionic operators [7]). In our case this means the following: given the commutation relation (1), one may *derive* the prescription (2) for normal ordering. Now assume that we did not know that the operators satisfy (1) - but we did know the prescription (2) for arbitrary words in  $b$  and  $b^\dagger$ . Could we conclude that (1) is satisfied? As a first step, we would find that  $\mathcal{N}([b, b^\dagger] - I) = 0$ . However, to be able to conclude that  $[b, b^\dagger] = I$ , we would have to make some further assumptions.

Let us now turn to the generalized normal ordering  $\mathcal{N}_p$ . Above we have only spoken about the letters  $a$  and  $a^\dagger$  (to which  $\mathcal{N}_p$  is applied) and have not interpreted these as bosonic operators. The reason for this is simple: the prescription (5) is not consistent with (1)! Since this is an important point we will discuss it explicitly. We consider the simplest case where - apart from the null contraction - only one contraction is involved, namely the word  $aa^\dagger$ . From the definition (5) it follows that

$$\mathcal{N}_p(aa^\dagger) = :aa^\dagger : + :pI := a^\dagger a + pI \quad (7)$$

(for ease of comparison we have written the identity explicitly). On the other hand, assuming that (1) holds we write  $aa^\dagger = a^\dagger a + I$  and - using linearity of  $\mathcal{N}_p$  - find that

$$\mathcal{N}_p(aa^\dagger) = \mathcal{N}_p(a^\dagger a + I) = \mathcal{N}_p(a^\dagger a) + \mathcal{N}_p(I) = a^\dagger a + I,$$

which clearly contradicts (for the case  $p \neq 1$  we are interested in) the result (7) obtained directly from the definition. This shows that *the letters  $a$  and  $a^\dagger$  - which are normal ordered according to*

$\mathcal{N}_p$  - cannot satisfy (1), i.e., they cannot be interpreted straightforwardly as bosonic annihilation and creation operators! Note that (7) can also be written as

$$\mathcal{N}_p([a, a^\dagger] - pI) = 0.$$

This calculation suggests that the effect of introducing the generalized normal ordering  $\mathcal{N}_p$  on the otherwise arbitrary letters  $a$  and  $a^\dagger$  might be equivalent to the conventional normal ordering  $\mathcal{N}$  where the letters satisfy the “generalized Heisenberg algebra”  $[a, a^\dagger] = pI$ . However, this cannot be the case since a rescaling of the letters by a factor  $\sqrt{p}^{-1}$  would result in the conventional Heisenberg algebra (1).

As another example consider the  $q$ -deformed boson operators  $a_q$  and  $a_q^\dagger$  satisfying  $a_q a_q^\dagger - q a_q^\dagger a_q = I$ , or

$$a_q a_q^\dagger = q a_q^\dagger a_q + I \quad (8)$$

(for combinatorial aspects of normal ordering  $q$ -deformed bosons see [8] and the references therein). Due to the appearing factor of  $q$  it is clear (same argument as above) that the generalized normal ordering  $\mathcal{N}_p$  cannot be equivalent to the standard normal ordering  $\mathcal{N}$ , where the letters satisfy the  $q$ -deformed Heisenberg algebra (8). Note that choosing  $q = -1$  in (8) yields the canonical anticommutation relations of fermionic annihilation and creation operators; thus  $\mathcal{N}_p$  is neither interpretable in terms of conventional fermionic annihilation and creation operators  $f$  and  $f^\dagger$  satisfying  $ff^\dagger + f^\dagger f = I$ .

Let us discuss another aspect of the letters  $a$  and  $a^\dagger$ . Suppose we want to represent them as operators in a Fock space. For this we first introduce the vacuum  $|0\rangle$  and define the states  $|n\rangle$  through the action of  $a^\dagger$  in the usual fashion, i.e.,  $|n\rangle := \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$ ; note that this implies  $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ . The Fock space is then the linear hull of the states  $|n\rangle$ . We may also define “coherent states” associated to  $\gamma \in \mathbb{C}$  in the usual manner, i.e.,

$$|\gamma\rangle = e^{-|\gamma|^2/2} \sum_{n \geq 0} \frac{\gamma^n}{\sqrt{n!}} |n\rangle. \quad (9)$$

But now comes the really difficult part: We have to define an action of  $a$  on the states  $|n\rangle$  (or  $|\gamma\rangle$ ) which is compatible with the generalized normal ordering  $\mathcal{N}_p$ . Assuming that  $a$  destroys one “quantum” we may write  $a|n\rangle = a_n|n-1\rangle$  and have to determine the coefficients  $a_n$  (we clearly have also  $a|0\rangle = 0$ ). Note that choosing  $a_n = \sqrt{n}$  yields the usual Fock representation, where we can derive  $a^\dagger a = N$  (where we define the number operator  $N$  by its action  $N|n\rangle = n|n\rangle$  on the states) as well as  $aa^\dagger = N + 1$  and, therefore,  $[a, a^\dagger] = I$ . As we have discussed above, this choice is not consistent with  $\mathcal{N}_p$ . Equivalently, if one introduces first  $N$  and demands  $N = a^\dagger a$  then one also finds  $a_n = \sqrt{n}$  and runs into the same problem. Thus, the definition of the action  $b$  on the states  $|n\rangle$  seems to be rather subtle. Note that if we want to follow the standard path we should also introduce the dual states  $\langle m|$  to be able to calculate expectation values and also have that  $a$  and  $a^\dagger$  are adjoint operators! Let us move to the action of  $a$  on the coherent states (9). Recall that in the standard bosonic case one of the defining properties of the coherent states is that they are eigenstates for the annihilation operator, i.e.,  $b|\gamma\rangle_b = \gamma|\gamma\rangle_b$  (here we have added a subscript “ $b$ ” to indicate that the coherent states are defined with the help of  $b^\dagger$ ). If we demand the analogous property for  $a$ , i.e.,  $a|\gamma\rangle = \gamma|\gamma\rangle$  - even without knowing the action of  $a$  on the  $|n\rangle$  - we are led to the conclusion  $a|n\rangle = \sqrt{n}|n-1\rangle$ . However, we have discussed above that this is not compatible with  $\mathcal{N}_p$ . Thus, it seems to be nontrivial to generalize the usual Fock representation (and the coherent states) of the bosonic operators  $b$  and  $b^\dagger$  satisfying (1) and  $\mathcal{N}$  to the letters  $a$  and  $a^\dagger$  satisfying  $\mathcal{N}_p$ .

To conclude, it does not seem to be straightforward to give the letters  $a$  and  $a^\dagger$  an interpretation as annihilation and creation operators of some sort of particles (which are definitely not conventional bosons since they do not satisfy (1)). This is the reason why we speak in the following about the *letters* (and not operators)  $a$  and  $a^\dagger$ . However, it would clearly be interesting to find out which algebraic

relations the letters  $a$  and  $a^\dagger$  can satisfy - as generalization of (1) - without contradicting (5); it would be even more interesting to find a relation from which (5) could be *derived* as consequence - in analogy to the conventional case where the usual normal ordering can be derived from the canonical commutation relation (cf. the discussion at the beginning of this section).

#### IV. AN EXPLICIT FORMULA FOR $\mathcal{N}_p((a^\dagger a)^n)$

The goal of this section to find a general formula for  $\mathcal{N}_p((a^\dagger a)^n)$ . First of all we need the following definition. Let  $\mathcal{F}_n$  to be the set of all vectors  $(\pi(1), \pi(2), \dots, \pi(n))$  of length  $n$  such that  $\pi(i) \in \{e, i, i+1, \dots, n\}$  and if  $\pi(i), \pi(j) \neq e$  then  $\pi(i) \neq \pi(j)$ . For example, the set  $\mathcal{F}_2$  consists of five vectors  $(e, e)$ ,  $(e, 2)$ ,  $(1, e)$ ,  $(2, e)$  and  $(1, 2)$ . We refine the set  $\mathcal{F}_n$  denoting by  $\mathcal{F}_{n,k}$  the set of all vectors in  $\mathcal{F}_n$  having exactly  $k$  coordinates  $e$ .

A *partition*  $\pi$  of  $[n] = \{1, 2, \dots, n\}$  is a collection  $P_1, P_2, \dots, P_k$  of nonempty disjoint subsets of  $[n]$ , called *blocks*, such that  $P_1 \cup P_2 \cup \dots \cup P_k = [n]$ . We may assume that  $P_1, P_2, \dots, P_k$  are listed in the increasing order of the blocks' cardinalities and we write  $P_1 < P_2 < \dots < P_k$ . The set of all partitions of  $[n]$  with  $k$  blocks is denoted by  $P_{n,k}$ . The cardinality of  $P_{n,k}$  is the well-known Stirling number  $S(n, k)$  of the second kind [10].

The next observation show that the elements of  $\mathcal{F}_{n-1,k-1}$  are enumerated by the Stirling number  $S(n, k)$ .

**Lemma 1** *There exists a bijection between the set  $\mathcal{F}_{n-1,k-1}$  and the set  $P_{n,k}$ . In particular, the cardinality of the set  $\mathcal{F}_{n-1,k-1}$  is given by  $S(n, k)$ , the Stirling number of the second kind.*

**Proof.** Let  $(\pi(1), \dots, \pi(n-1))$  be any vector in  $\mathcal{F}_{n-1,k-1}$  and let  $\pi$  be the vector  $(\pi(1), \dots, \pi(n-1), e)$ . We define the first block of the partition of  $[n]$  by

$$P_1 = \{\beta_{11} = 1, \beta_{12} = 1 + \pi(\beta_{11}), \dots, \beta_{1\ell} = 1 + \pi(\beta_{1(\ell-1)})\},$$

where  $\pi(\beta_{1\ell}) = e$ , and we define the  $i$ -th,  $i = 1, 2, \dots, k$ , block of the partition of  $[n]$  by

$$P_i = \{\beta_{i1} = a_i, \beta_{i2} = 1 + \pi(\beta_{i1}), \dots, \beta_{i\ell} = 1 + \pi(\beta_{i(\ell-1)})\},$$

where  $\pi(\beta_{i\ell}) = e$  and  $a_i$  is the minimal coordinate of the vector  $\pi$  such that  $a_i \notin P_j$  for all  $j = 1, 2, \dots, i-1$ . For instance, if  $\pi = (1, e, 3, e, e)$  then  $P_1 = \{1, 2\}$ ,  $P_2 = \{3, 4\}$ , and  $P_3 = \{5\}$ . From the above construction, we see that  $(\pi(1), \dots, \pi(n-1))$  is a vector in  $\mathcal{F}_{n-1,k-1}$  if and only if  $P_1, P_2, \dots, P_k$  is a partition of  $[n]$ , as required. ■

For example, Lemma 1 for  $n = 2$  gives

$$\rho(ee) = \{1\}, \{2\}, \{3\}; \rho(e2) = \{1\}, \{23\}; \rho(1e) = \{12\}, \{3\}; \rho(12) = \{123\}; \rho(2e) = \{13\}, \{2\}.$$

Now we prove a bijection between the set of contractions of  $(a^\dagger a)^n$  and the set  $\mathcal{F}_n$ . To do that we need the following definition. We say that a contraction  $\pi \in \mathcal{C}((a^\dagger a)^n)$  has *degree*  $k$  if  $\pi := (a^\dagger)^v a^w p^k$  for  $v, w \geq 0$ . The set of the contractions in  $\mathcal{C}((a^\dagger a)^n)$  having degree  $k$  is denoted by  $C_{n,k}$ .

**Lemma 2** *There exists a bijection between the set of contractions  $C_{n,k-1}$  and the set  $\mathcal{F}_{n-1,k-1}$ .*

**Proof.** Let  $w = w_{2n}w_{2n-1} \dots w_1$  be any contraction of  $(a^\dagger a)^n$ . For each  $j = 1, 2, \dots, n-1$ , define  $\pi(j) = e$  if  $w_{2j} = a^\dagger$ , and  $\pi(j) = i$  if  $w_{2j} = e^\dagger$  and  $w_{2i+1} = e$ , where  $i$  is minimal and greater than  $j$ . The definition of  $\pi$  implies that  $w$  is a contraction of  $(a^\dagger a)^n$  if and only if the vector  $\pi = (\pi(1), \pi(2), \dots, \pi(n-1)) \in \mathcal{F}_{n-1}$ . Moreover, if  $w$  is a contraction then the vector  $\pi$  has  $k-1$  coordinates  $e$  if and only if  $k-1 = \#\{j | w_{2j} = a^\dagger\}$ , or, in other words,  $\pi \in \mathcal{F}_{n-1,k-1}$  if and only if  $w \in C_{n,k-1}$ . ■

Let  $Q$  be any subset of  $[n] = \{1, 2, \dots, n\}$ . We say that  $Q$  has a *rise* at  $i$  if  $i, i+1 \in Q$ . The number of rises of  $Q$  is denoted by  $\text{rise}(Q)$ . Let  $\pi = P_1, P_2, \dots, P_k$  be any partition of  $[n]$ , we define  $\text{rise}(\pi) = \sum_{j=1}^k \text{rise}(P_j)$  and we say that  $\pi$  has exactly  $\text{rise}(\pi)$  rises. Lemma 1 together with Lemma 2 give then the following result.

**Proposition 3** *There exists a bijection between the set of contractions  $C_{n,k}$  and the set of partitions of  $n$  with  $k$  rises.*

We are now ready to give a formula for  $\mathcal{N}_p((a^\dagger a)^n)$ .

**Theorem 4** *For all  $n \geq 1$ ,*

$$\mathcal{N}_p((a^\dagger a)^n) = \sum_{k=0}^n S_p(n, k) (a^\dagger)^k a^k, \quad (10)$$

where  $S_p(n, k)$  satisfies the following recurrence relation

$$S_p(n, k) = (k-1+p)S_p(n-1, k) + S_p(n-1, k-1),$$

with the initial conditions  $S_p(n, 1) = p^{n-1}$  and  $S_p(n, k) = 0$  for all  $k > n$ .

**Proof.** Denote by  $S(n, k; m)$  the number of contractions  $\pi$  of  $(a^\dagger a)^n$  such that  $\pi := (a^\dagger)^k a^k p^m$ . From Proposition 3, we get that  $S(n, k; m)$  is the number of partitions of  $[n]$  into  $k$  blocks  $P_1, \dots, P_k$  with  $m$  rises. To find a recurrence relation for the sequence  $S(n, k; m)$ , we consider the position of  $n$  in the blocks  $P_1, \dots, P_k$  with  $P_1 < P_2 < \dots < P_k$ . If  $P_k = \{n\}$  then there are  $S(n-1, k-1; m)$  such partitions; if  $n \in P_i$  and  $n-1 \notin P_i$  then there are  $S(n-1, k; m)$  partitions; if  $n, n-1 \in P_i$  then the number of partitions is  $S(n-1, k; m-1)$ . Therefore,

$$S(n, k; m) = (k-1)S(n-1, k; m) + S(n-1, k; m-1) + S(n-1, k-1; m),$$

which is equivalent to

$$S_p(n, k) = (k-1+p)S_p(n-1, k) + S_p(n-1, k-1),$$

where  $S_p(n, k) = \sum_{m=0}^{n-k} S(n, k; m) p^m$ . Hence,

$$\mathcal{N}_p((a^\dagger a)^n) = \sum_{k=0}^n \sum_{m=0}^{n-k} S(n, k; m) p^m (a^\dagger)^k a^k = \sum_{k=0}^n S_p(n, k) (a^\dagger)^k a^k.$$

The initial conditions can be checked directly from the definitions. ■

As we mentioned above the polynomials  $S_p(n, k)$  satisfy the recurrence relation

$$S_p(n, k) = (k-1+p)S_p(n-1, k) + S_p(n-1, k-1),$$

with the initial conditions  $S_p(n, 1) = p^{n-1}$  and  $S_p(n, k) = 0$  for all  $k > n$ . Considering  $p = 1$  yields the recurrence relation (4) of the conventional Stirling numbers  $S_1(n, k) \equiv S(n, k)$ . If we define

$$S_p(x; k) = \sum_{n \geq k} S_p(n, k) \frac{x^n}{n!},$$

it is not hard to see that

$$S_p(x; k) = \frac{(p-1)!}{(p+k-1)!} + \frac{e^{px}}{(k-1)!} \sum_{j=0}^{k-1} \frac{(-1)^{k-1-j} \binom{k-1}{j}}{p+j} e^{jx} = \int_0^x \frac{e^{pt}}{(k-1)!} (e^t - 1)^{k-1} dt.$$

Thus the generating function

$$S_p(x, y) = \sum_{n \geq 1} \sum_{k=1}^n S_p(n, k) y^k \frac{x^n}{n!}$$

is given by

$$S_p(x, y) = \int_0^x y e^{pt} e^{y(e^t-1)} dt.$$

Notice that

$$S_1(x, y) = e^{y(e^x-1)}$$

is the generating function for the Stirling numbers of the second kind. One of the first explicit expressions for the numbers  $S_p(n, k)$  was given by d'Ocagne [11]:

$$S_p(n, k) = \frac{(-1)^{k-1}}{(k-1)!} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} (p+j)^{n-1},$$

for all  $n \geq k \geq 1$ . With this expression, Theorem 4 gives an explicit formula for  $\mathcal{N}_p((a^\dagger a)^n)$ .

**Theorem 5** *For all  $n \geq 1$ ,*

$$\mathcal{N}_p((a^\dagger a)^n) = \sum_{k=0}^n \left( \frac{(-1)^{k-1}}{(k-1)!} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} (p+j)^{n-1} \right) (a^\dagger)^k a^k.$$

Theorem 5 gives equation (3) for  $p = 1$ . For  $p = 0$ , Theorem 4 gives

$$\mathcal{N}_0((a^\dagger a)^n) = \sum_{j=0}^n S_0(n, j) (a^\dagger)^j a^j,$$

where  $S_0(n, k)$  satisfies the following recurrence relation  $S_0(n, k) = (k-1)S_0(n-1, k) + S_0(n-1, k-1)$  with initial condition  $S_0(n, 1) = \delta_{n,1}$  and  $S_0(n, k) = 0$ , for  $k > n$ . Thus,  $\sum_{k=1}^n S_0(n, k) = B_{n-1}$ , the  $(n-1)$ -th Bell number.

## V. A GENERAL CLASS OF NORMAL ORDERINGS

In this section we want to outline a rather general approach to generalizations of the normal ordering using contractions and the double dot operation. As in the definition of  $\mathcal{N}_p$  in Section II, we consider words  $F(a, a^\dagger)$  in the letters  $a$  and  $a^\dagger$  (which satisfy no relation). The set of *resulting contractions* - in the following abbreviated by *r-contractions* - of  $F(a, a^\dagger)$  will be denoted by  $\mathcal{RC}(F(a, a^\dagger))$  and it is the multiset of all words which result by pairing the letters  $a^\dagger$  with the letters  $a$  and omitting them, see Figure 1. For example, if  $F(a, a^\dagger) = aa^\dagger aa^\dagger$  then  $\mathcal{RC}(aa^\dagger aa^\dagger) = \{aa^\dagger aa^\dagger, aa^\dagger, aa^\dagger, 1, 1\}$ , where the first r-contraction comes from the null contraction, the next r-contraction results by pairing the first  $a$  with the first  $a^\dagger$ , the next r-contraction results by pairing the first  $a$  with the second  $a^\dagger$ , the next r-contraction results by pairing the second  $a$  with the second  $a^\dagger$ , the next r-contraction results by pairing the first  $a$  with the first  $a^\dagger$  and the second  $a$  with the second  $a^\dagger$ , and the final r-contraction results by pairing the first  $a$  with the second  $a^\dagger$  and the second  $a$  with the first  $a^\dagger$ .

Since each r-contraction  $\pi$  contains only the letters  $a$  and  $a^\dagger$  we can define  $\pi$  as in the conventional case by arranging the letters so that all  $a^\dagger$  precede all  $a$  (together with  $1 := 1$ ). Now, assume that we have a prescription associating to each  $\pi \in \mathcal{RC}(F(a, a^\dagger))$  a *weight*  $\mathcal{W}(\pi)$  (concrete examples will follow soon). Then we can define the  $\mathcal{W}$ -generalized normal ordering by

$$\mathcal{N}_{\mathcal{W}}(F(a, a^\dagger)) := \sum_{\pi \in \mathcal{RC}(F(a, a^\dagger))} \mathcal{W}(\pi) : \pi : . \quad (11)$$

This generalizes the normal ordering (2) and also (5). Note that there exists a bijection between contractions and r-contractions; however, in our above definition of contractions in Section II we have implicitly built in the weights. To separate these aspects we have introduced here in the general case the r-contractions. Clearly, if the weights of all r-contractions are equal to one then (11) reduces to (2) and one may identify contractions with r-contractions.

Let us define the *r-degree* of a r-contraction as the number of pairs of letters  $a$  and  $a^\dagger$  which have been contracted in the original word. We furthermore define a *crossing* of a contraction as in [12], i.e., if  $e = (i, j)$  and  $f = (k, l)$  are two edges in the word  $F(a, a^\dagger)$  then we say that  $e$  *crosses*  $f$  if they intersect with each other, i.e., if  $i < k < j < l$  or  $k < i < l < j$  (e.g., only the last contraction of Figure 1 is crossing). A contraction having no crossings is called *noncrossing*. Clearly, all contractions of degree less than or equal to one are noncrossing. The *crossing number*  $\mathcal{X}(\pi)$  of a contraction  $\pi$  is defined to be the number of crossings and the crossing number of a r-contraction is defined to be the crossing number of the associated contraction.

For the general class of normal orderings we are interested in we require the weights of the r-contractions to satisfy the following properties:

1. The weight is the product of a *crossing weight* and a *contraction weight*, where the crossing weight only depends on the number of crossings and the contraction weight does not depend on the number of crossings.
2. The contraction weight  $\mathcal{CW}$  of a r-contraction of r-degree one depends only on the distance of the letters which are contracted. The contraction weight of a r-contraction of r-degree  $k$  is the product of the weights of the  $k$  “subcontractions” it consists of.
3. The crossing weight of a r-contraction  $\pi$  is given by  $q^{\mathcal{X}(\pi)}$ , where  $q$  is the crossing weight of a r-contraction with one crossing.

To formalize these rules we introduce an infinite sequence  $\omega$  of weights, that is  $\omega = (\omega_{-1}, \omega_0 = 1, \omega_1, \omega_2, \dots)$ . The elements have the following interpretation:  $\omega_{-1}$  is the crossing weight for one crossing denoted by  $q$  above,  $\omega_0 = 1$  is the weight of the null contraction and  $\omega_n$  with  $n \geq 1$  are the contraction weights for a r-contraction of degree one where the pair of contracted letters has distance  $n$ . We are now ready to define the  $\omega$ -generalized normal ordering by letting

$$\mathcal{N}_{\omega}(F(a, a^\dagger)) := \sum_{\pi \in \mathcal{RC}(F(a, a^\dagger))} \omega_{-1}^{\mathcal{X}(\pi)} \mathcal{CW}_{\omega}(\pi) : \pi : . \quad (12)$$

Comparing (12) with (11) shows that the weight of a r-contraction  $\pi$  is given in terms of  $\omega$  by  $\omega_{-1}^{\mathcal{X}(\pi)} \mathcal{CW}_{\omega}(\pi)$ . Before giving explicit examples of the sequence  $\omega$  we consider some concrete examples of the  $\omega$ -generalized normal ordering. As a simple example one has  $\mathcal{N}_{\omega}((a^\dagger a)^2) = \omega_0 (a^\dagger)^2 a^2 + \omega_1 a^\dagger a$  (where we have explicitly denoted the weight of the null contraction with  $\omega_0 = 1$ ). A slightly more involved example is given by

$$\mathcal{N}_{\omega}((a^\dagger a)^3) = \omega_0 (a^\dagger)^3 a^3 + \{2\omega_1 + \omega_3\} (a^\dagger)^2 a^2 + \omega_1^2 a^\dagger a. \quad (13)$$



Clearly, if the sequence  $\omega$  is given by  $\omega_1 = p$  and  $\omega_i = 1$  for  $i \neq 1$  then (13) reduces to (6). Note that up to now no r-contractions with crossings have appeared. In the next example there appears one crossing (when the first  $a$  is contracted with the third  $a^\dagger$  and the second  $a$  with the fourth  $a^\dagger$ ):

$$\mathcal{N}_\omega((a^\dagger a)^4) = \omega_0(a^\dagger)^4 a^4 + \{3\omega_1 + 2\omega_3 + \omega_5\}(a^\dagger)^3 a^3 + \{2\omega_1^2 + 2\omega_1\omega_3 + \omega_{-1}\omega_3^2 + \omega_1\omega_5\}(a^\dagger)^2 a^2 + \omega_1^3 a^\dagger a. \quad (14)$$

Let us now consider some special sequences  $\omega$ :

1.  $\omega^{(1)} = (1, 1, 1, 1, \dots)$ : The weight of all r-contractions is one (independent of crossings). This corresponds to the conventional normal ordering, i.e.,  $\mathcal{N}_{\omega^{(1)}} = \mathcal{N}_1 = \mathcal{N}$ .
2.  $\omega^{(p)} = (1, 1, p, 1, \dots)$ : Since  $\omega_{-1} = 1$  crossings play no role. Only  $\omega_1 = p$  is not equal to one, thus all r-contractions of degree one have weight one except those where the contracted operators are adjacent - then the r-contraction has weight  $p$ . This reproduces exactly the generalized normal ordering  $\mathcal{N}_p$  from above, i.e.,  $\mathcal{N}_{\omega^{(p)}} = \mathcal{N}_p$ .
3.  $\omega^{nc} = (0, 1, 1, 1, \dots)$ : All r-contractions have weight one, except those, where at least one crossing occurs - these r-contractions have weight zero. Thus, this example yields the *noncrossing normal ordering* considered in [12].

Returning to the general case, one may consider again the particular word  $F(a, a^\dagger) = (a^\dagger a)^n$ . Denoting the set of r-contractions of r-degree  $l$  by  $\mathcal{RC}^{(l)}((a^\dagger a)^n)$ , we have the disjoint union  $\mathcal{RC}((a^\dagger a)^n) = \bigcup_{l=0}^n \mathcal{RC}^{(l)}((a^\dagger a)^n)$ . Note that for any  $\pi \in \mathcal{RC}^{(l)}((a^\dagger a)^n)$  there have been  $l$  pairs of letters  $a$  and  $a^\dagger$  contracted, leaving  $n - l$  letters  $a$  and  $a^\dagger$ . Thus,  $\pi \in \mathcal{RC}^{(l)}((a^\dagger a)^n) \Rightarrow: \pi := (a^\dagger)^{n-l} a^{n-l}$ . Combining these results yields

$$\mathcal{N}_\omega((a^\dagger a)^n) = \sum_{l=0}^n \left\{ \sum_{\pi \in \mathcal{RC}^{(l)}((a^\dagger a)^n)} \omega_{-1}^{\mathcal{X}(\pi)} \mathcal{CW}_\omega(\pi) \right\} (a^\dagger)^{n-l} a^{n-l}. \quad (15)$$

Thus, introducing the  $\omega$ -generalized Stirling numbers of second kind by

$$S_\omega(n, k) := \sum_{\pi \in \mathcal{RC}^{(n-k)}((a^\dagger a)^n)} \omega_{-1}^{\mathcal{X}(\pi)} \mathcal{CW}_\omega(\pi), \quad (16)$$

we can write (15) in analogy to (3) and (10) as

$$\mathcal{N}_\omega((a^\dagger a)^n) = \sum_{k=1}^n S_\omega(n, k) (a^\dagger)^k a^k. \quad (17)$$

The usual Stirling numbers are obtained if all weights are equal to one, i.e.,  $\omega = \omega^{(1)}$  from above:

$$S(n, k) = S_{\omega^{(1)}}(n, k) = |\mathcal{RC}^{(n-k)}((a^\dagger a)^n)|.$$

If  $\omega = \omega^{(p)}$  from above then  $S_{\omega^{(p)}}(n, k) = S_p(n, k)$ . For the general Stirling numbers (16) it might be interesting to obtain some general results (e.g., explicit values, recurrence relation). Let us mention some simple observations. Since there exists exactly one null contraction,  $S_\omega(n, n) = \omega_0 = 1$ . On the other hand, there exists exactly one r-contraction of r-degree  $n - 1$ , where all letters except the “boundaries” are contracted (thus, all adjacent pairs  $aa^\dagger$  are contracted), implying  $S_\omega(n, 1) = \omega_1^{n-1}$ . These two values can be seen nicely in the above explicit examples (13) and (14). Of course, one may define in analogy to the usual case also  $\omega$ -generalized Bell numbers by  $B_\omega(n) = \sum_{k=0}^n S_\omega(n, k)$ .

## VI. CONCLUSION

We have generalized results of [1] by defining a *refined* version of the normal ordering, and we have discussed the physical aspects of the approach considered. The essence of the generalization  $\mathcal{N}_p$  can be described as follows. For a given expression  $F(a^\dagger, a)$ , we use the set of all contractions in order to get the normally ordered form  $\mathcal{N}(F(a^\dagger, a))$ . Each contraction can be seen as having weight one (that is all contractions are equidistributed). We associate a weight to each contraction (corresponding to the number of  $ee^\dagger$ 's in the expression). In this way, we can study the normally ordered form for the given expression based on the set of contractions with a given weight.

Several authors studied the standard normally ordered form of different kind of expressions, such as  $(a^\dagger a)^n$ ,  $(a^r + a^\dagger)^n$  and  $(a + (a^\dagger)^r)^n$  [2]. Putting on the side a potential operational value, the form  $\mathcal{N}_p(F(a, a^\dagger))$  is interesting because it says more than the standard normally ordered form  $\mathcal{N}_1(F(a, a^\dagger))$ . Indeed, this generalization implies extra information on the set of the contractions of a given expression  $F(a, a^\dagger)$ . Additionally, we have introduced a further generalization, where contractions between vertices of different distances are allowed to have arbitrary weights.

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